NOTE ON THE ELECTROMAGNETIC THEORY OF LIGHT

1. The statement of the electromagnetic theory of light in my former paper was connected with several other electromagnetic investigations, and was therefore not easily understood when taken by itself. I propose, therefore, to state it in what I think the simplest form, deducing it from admitted facts, and showing the connexion between the experiments already described and those which determine the velocity of light.

[Four Electromagnetic Theorems]

2. The connexion of electromagnetic phenomena may be stated in the following manner.

3. Theorem A.—If a closed curve be drawn embracing an electric current, then the integral of the magnetic intensity taken round the closed curve is equal to the current multiplied by 4π.

4. The integral of the magnetic intensity may be otherwise defined as the work done on a unit magnetic pole carried completely round the closed curve.

5. This well-known theorem gives us the means of discovering the position and magnitude of electric currents, when we can ascertain the distribution of magnetic force in the field. It follows directly from the discovery of ŒRSTED.

This "Note" was appended to Maxwell's 1868 paper, *On a Method of Making a Direct Comparison of Electrostatic with Electromagnetic Force*, the "former paper" referred to in Maxwell's opening sentence. Besides presenting a theory which is of immense importance in itself, it advances our path to the modern Maxwell equations by introducing theorems which will lead directly to those equations. I have altered Maxwell's notation for greater consistency with his nomenclature in the *Treatise* and have also supplied paragraph numbers to facilitate commentary.

3. *Theorem A*: Compare *Treatise* (479.), where Maxwell stated the equivalent of $Hr = 2i$ for the magnetic intensity H at distance r from a long straight wire carrying current i. Theorem A is the basis for one of the four equations universally called "Maxwell's" today.

the integral of the magnetic intensity taken round the closed curve: That is, the line-integral around the curve. For the special case of a closed circular path of radius r, $H = \frac{2i}{r}$ is constant; so $\int H\,ds = \frac{2i}{r}\int ds = \frac{2i}{r}\cdot 2\pi r = 4\pi i$. This calculation can readily be extended to a closed path of any shape, with the same result.

6. Theorem B.—If a conducting circuit embraces a number of lines of magnetic force, and if, from any cause whatever, the number of these lines is diminished, an electromotive force will act round the circuit, the total amount of which will be equal to the decrement of the number of lines of magnetic force in unit of time.

7. The number of lines of magnetic force may be otherwise defined as the integral of the magnetic intensity resolved perpendicular to the surface, multiplied by the element of surface, and by the coefficient of magnetic induction, the integration being extended over any surface bounded by the conducting circuit.

8. This theorem is due to FARADAY, as the discoverer both of the facts and of this mode of expressing them, which I think the simplest and most comprehensive.

9. Theorem C.—When a dielectric is acted on by electromotive force it experiences what we may call electric polarization. If the direction of the electromotive force is called positive, and if we suppose the dielectric bounded by two conductors, A on the negative, and B on the positive

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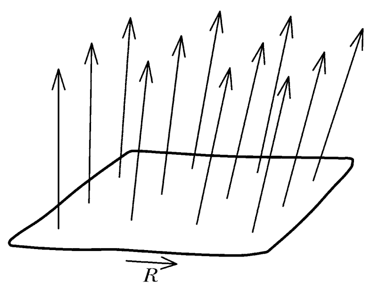
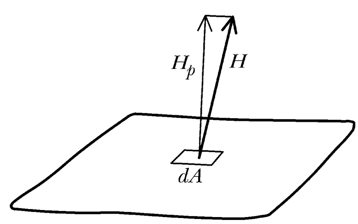
6. *Theorem B*: Compare *Treatise* (531.); it is the basis for another of the four equations called "Maxwell's" today. Note that the several lines embraced by the conducting circuit may have any direction.



electromotive force: The line-integral of electromotive intensity R along a path (*Treatise*, 69.). Algebraically expressed, $E = \int R \cos\varepsilon \, ds$, where E denotes the electromotive force.

7. *The number of lines of magnetic force may be otherwise defined*: In the *Treatise* (404.7) Maxwell gave reasons for reinterpreting Faraday's "lines of force" as *lines of induction*, defined by the quantity B rather than H. Maxwell is about to do so here.

the magnetic intensity resolved perpendicular to the surface: That is, the *perpendicular component* of magnetic intensity H with respect to the surface enclosed by the conducting circuit. In the sketch, H_p is the perpendicular component and dA is an infinitesimal element of that surface.



The magnetic induction B is a measure of the number of lines per unit perpendicular area (*Treatise* 489.2). Since $B = \mu H$ (*Treatise* 428.21, *comment*), the number of lines will equal $\mu H_p A$ for a perpendicular area A, or $\int \mu H_p \, dA$ for an irregular area whose perpendicular elements are dA. The "decrement" of this number in unit of time (6.) is its time rate of decrease. Theorem B may therefore be expressed algebraically as $E = -\frac{d}{dt}\int \mu H_p \, dA$.

side, then the surface of the conductor A is positively electrified, and that of B negatively. If we admit that the energy of the system so electrified resides in the polarized dielectric, we must also admit that within the dielectric there is a displacement of electricity in the direction of the electromotive force, the amount of this displacement being proportional to the electromotive force at each point, and depending also on the nature of the dielectric.

10. The energy stored up in any portion of the dielectric is half the product of the electromotive force and the electric displacement, multiplied by the volume of that portion.

11. It may also be shown that at every point of the dielectric there is a mechanical tension along the lines of electric force, combined with an equal pressure in all directions at right angles to these lines, the amount of tension on unit of area being equal to the amount of energy in unit of volume.

12. I think that these statements are an accurate rendering of the ideas of FARADAY, as developed in various parts of his 'Experimental Researches.'

13. Theorem D.—When the electric displacement increases or diminishes, the effect is equivalent to that of an electric current in the positive or negative direction.

14. Thus, if the two conductors in the last case are now joined by a wire, there will be a current in the wire from A to B.

15. At the same time, since the electric displacement in the dielectric is diminishing, the will be an action electromagnetically equivalent to that of an electric current from B to A through the dielectric.

16. According to this view, the current produced in discharging a condenser is a complete circuit, and might be traced within the dielectric

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9. *Theorem C. The surface of conductor A is positively electrified, and that of B negatively*: Compare *Treatise* (62.6).

within the dielectric there is a displacement of electricity … proportional to the electromotive force: Compare *Treatise* (68.6).

10. *The energy … is half the product…*: This relation is derived in Appendix 2.

11. *there is a mechanical tension*: In the complete Treatise, Maxwell showed that a dielectric in which electric displacement takes place will experience a mechanical tension very much like the tension in a taut string. While the existence of such tension is not important for his present argument, it is an essential element in Maxwell's claim to be accurately rendering Faraday's views (12).

13. *Theorem D. The effect is equivalent to that of an electric current*: compare *Treatise* (60.5).

14. *there will be a current in the wire from A to B*: Compare *Treatise* (60.14) and comment.

itself by a galvanometer properly constructed. I am not aware that this has been done, so that this part of the theory, though apparently a natural consequence of the former, has not been verified by direct experiment. The experiment would certainly be a very delicate and difficult one.

[Electromagnetic Theory of Light]

17. Let us now apply these four principles to the electromagnetic theory of light, considered as a disturbance propagated in plane waves.

18. Let the direction of propagation be taken as the axis of z, and let all the quantities be functions of z and of t the time; that is, let every portion of any plane perpendicular to z be in the same condition at the same instant.

19. Let us also suppose that the magnetic force is in the direction of the axis of y, and let H be the magnetic intensity in that direction at any point.

20. Let the closed curve of Theorem A consist of a parallelogram in the plane yz, two of whose sides are b along the axis of y, and z along the axis of z. The integral of the magnetic intensity taken round this parallelogram is $b(H_0 - H)$ where H_0 is the value of H at the origin.

21. Now let j be the quantity of electric current in the direction of x per unit of area taken at any point, then the whole current through the parallelogram will be

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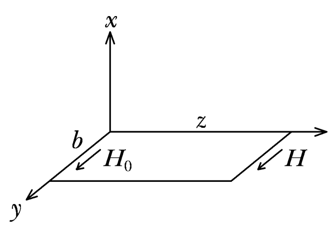
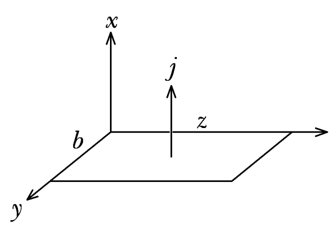
17. *plane waves*: waves in which all points of any plane perpendicular to the direction of propagation are in the same condition at the same instant of time; alternatively stated, they are *in phase*. Maxwell will affirm this property explicitly in the next paragraph.

20. *Let the closed curve ... consist of a parallelogram in the plane yz*: Each of the accompanying diagrams represents a "snapshot" of some region of space at a particular instant in time.



The integral of the magnetic intensity ... is $b(H_0 - H)$: The positive direction of rotation in the yz-plane is from the y-axis to the z-axis (*Treatise* 23.). Proceeding in that direction around the parallelogram, the successive products of magnetic intensity and distance add up to

$$H_0 b + 0 - Hb - 0 = b\left(H_0 - H\right).$$



21. *the whole current*: That is, the current that passes through the whole area bz. If current density j were constant, it would only be necessary to multiply j by that area; but since j may vary along the z-direction, we must integrate with respect to z. We do not have to integrate with respect to y because all points in any xy plane are in the same condition (18.); hence j does not vary in the y-direction.

$$\int_0^z bj\,dz,$$

and we have by (A),

$$b\left(H_0 - H\right) = 4\pi \int_0^z bj\,dz.$$

22. If we divide by b and differentiate with respect to z, we find

$$\frac{dH}{dz} = -4\pi j. \qquad (14)$$

23. Let us next consider a parallelogram in the plane of xz, two of whose sides are a along the axis of x, and z along the axis of z.

24. If R is the electromotive force per unit length in the direction of x, then the total electromotive force round this parallelogram is $a(R - R_0)$.

25. If μ is the coefficient of magnetic induction, then the number of lines of force embraced by this parallelogram will be

$$\int_0^z a\mu H\,dz,$$

and since by (B) the total electromotive force is equal to the rate of diminution of the number of lines in unit of time,

$$a\left(R - R_0\right) = -\frac{d}{dt}\int_0^z a\mu H\,dz.$$

26. Dividing by a and differentiating with respect to z, we find

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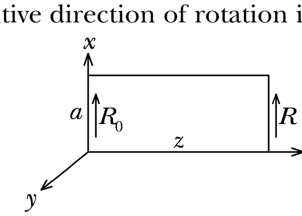
22. *Equation* (14): The negative sign results from differentiating both sides, as Maxwell states. Its presence means that if the overall change of H with respect to z is, say, positive (H greater than H_0), the current density j within the rectangle will be negative; in the drawing opposite that is *downwards*, the negative direction of x.

23. *Let us next consider a parallelogram in the plane of xz*: See the diagram below.

24. *R is the electromotive force per unit length*: Maxwell ordinarily calls R the electromotive intensity. But electromotive force is the line-integral of R with respect to distance (*Treatise*, 69.1), so the alternative name is appropriate.

the total electromotive force ... is $a(R-R_0)$: The positive direction of rotation in the xz-plane is from the z-axis to the x-axis (*Treatise* 23.). Proceeding in that direction around the parallelogram, the successive products of electromotive intensity and distance add up to

$$0 + Ra - 0 - R_0 a = a\left(R - R_0\right).$$



25. *the number of lines of force embraced by the parallelogram*: Refer to the integral $\int \mu H_p\,dA$ cited in (7., *comment*), and there set $dA = a\,dz$ to obtain Maxwell's expression. The "rate of diminution" of this quantity is its negative derivative with respect to time (since its *positive* derivative would be its rate of *increase*).

$$\frac{dR}{dz} = -\mu\frac{dH}{dt} \tag{15}$$

27. Let the nature of the dielectric be such that an electric displacement D is produced by an electromotive force R,

$$R = kD \tag{16}$$

where k is a quantity depending on the particular dielectric, which may be called its "electric elasticity."

28. Finally, let the current j, already considered, be supposed entirely due to the variation of D, the electric displacement, then

$$j = \frac{dD}{dt} \tag{17}$$

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27. *electromotive force R*: More correctly, electromotive force *per unit charge*, that is, electromotive intensity. Maxwell has consistently used the symbol R to represent electromotive intensity.

$R = kD$: Compare *Treatise* (62.3), where Maxwell stated the prose equivalent of $D = \frac{K}{4\pi}R$ or $R = \frac{4\pi}{K}D$. Recall that $k = 4\pi/K$, as noted earlier in the comments on (19.2) of the previous reading, page 143 above.

28. *let the current j … be supposed entirely due to the variation of D*: Except for sparks, there is no conduction in empty space; any current must be due solely to changing displacement D.

Equation (17): Note that both j (current density) and D (displacement) are quantities defined *per unit area.*

29. *eliminate j, R, and D*: Eliminate j between equations (14) and (17) to obtain

$$\frac{dH}{dz} = -4\pi\frac{dD}{dt} .$$

Next eliminate D between this result and equation (16), which yields

$$\frac{dH}{dz} = -\frac{4\pi}{k}\frac{dR}{dt} .$$

Differentiate both sides of this expression with respect to z; if we may assume that the order of differentiation makes no difference, this yields

$$\frac{d^2H}{dz^2} = -\frac{4\pi}{k}\frac{d}{dz}\left(\frac{dR}{dt}\right) = -\frac{4\pi}{k}\frac{d}{dt}\left(\frac{dR}{dz}\right).$$

Now eliminate dR/dz between this result and equation (15) to obtain

$$\frac{d^2H}{dz^2} = \frac{4\pi\mu}{k}\frac{d}{dt}\left(\frac{dH}{dt}\right) = \frac{4\pi\mu}{k}\frac{d^2H}{dt^2} .$$

Multiply through by $k/4\pi\mu$ to obtain the desired equation. With a different set of eliminations one can describe a similar equation in R instead of H.

[The Electromagnetic Wave Equation]

29. We have now four equations, (14), (15), (16), (17), between the four quantities H, j, R, D. If we eliminate j, R, and D, we find

$$\frac{d^2H}{dt^2}=\frac{k}{4\pi\mu}\frac{d^2H}{dz^2}. \qquad (18)$$

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Equation (18): It is a *wave equation*, since the second derivative of a quantity (in this case, H) with respect to time is proportional to the second derivative of the same quantity with respect to distance: see Appendix 5. It describes a wave traveling in the z-direction with velocity equal to $\sqrt{k/4\pi\mu}$, or $\sqrt{k/4\pi}$ in vacuum, where $\mu = 1$ (in the electromagnetic system).

But in equation (46) of the previous reading, page 145 above, Maxwell showed that $k = 4\pi c^2$. Substituting this expression, we find for the wave velocity

$$\sqrt{\frac{k}{4\pi}} = \sqrt{\frac{4\pi c^2}{4\pi}} = c\ .$$

It is thus clear that c, the ratio of the electromagnetic to the electrostatic unit of charge, is the value of a real velocity—and must therefore have the dimensions of a velocity, as Maxwell stated in *A Dynamical Theory* (19.2), page 143 above. Furthermore, the value of c had already been measured experimentally; and, as Maxwell wrote elsewhere in the same paper, its value proved to be "so nearly that of light, that it seems we have strong reason to conclude that light itself ... is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws."