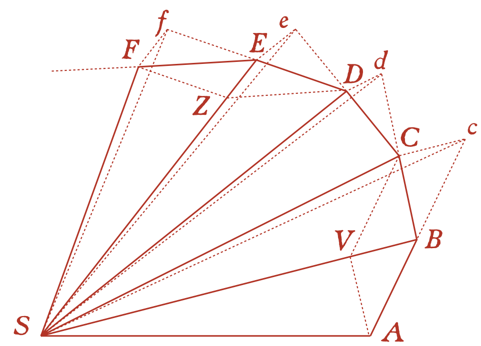
SECTION 2

On the Finding of Centripetal Forces

Proposition 1

The areas which bodies driven in orbits [gyros] *describe by radii drawn to an immobile center of forces, are contained in immobile planes and are proportional to the times.*

Let the time be divided into equal parts, and in the first part of the time let the body, by its inherent force, describe the straight line AB. In the second part of the time, the same body, if nothing were to impede it, would pass on by means of a straight line to c (by Law 1), describing the line Bc equal to AB, with the result that, radii AS, BS, cS being drawn to the center, the areas ASB, BSc would come out equal. But when the body comes to B, let the centripetal force act with an impulse that is single but great, and let it have the effect of making the body depart from the straight line Bc and continue in the straight line BC. Let cC be drawn parallel to BS, meeting BC at C; and, the second part of the time being completed, the body (by Corollary 1 of the Laws) will be located at C, in the same plane as the triangle ASB.



Connect SC, and, because of the parallels SB, Cc, triangle SBC will be equal to triangle SBc, and therefore also to triangle SAB. By a similar argument if the centripetal force should act successively at C, D, E, and so on, making the body describe the individual straight lines CD, DE, EF, and so on, in the individual particles of time, all these will lie in the same plane, and the triangle SCD will be equal to the triangle SBC, and SDE to SCD, and SEF to SDE. Therefore, in equal times equal areas are described in a motionless plane; and, *componendo*, any sums whatever of areas $SADS$, $SAFS$ are to one another as are the times of description. Now let the number of the triangles be increased and their breadth decreased *in infinitum*, and their ultimate perimeter ADF (by Corollary Four of Lemma Three) will be a curved line: and therefore the centripetal force, by which the body is perpetually drawn back from the tangent of this curve, will act without ceasing, while any described areas whatever $SADS$, $SAFS$, always proportional to the times of description, will be proportional to those same times in this case.

Q.E.D.

Notes on Book One Proposition 1

One stands in awe of the amazing power of this simple proposition, on which rests all of Newton's celestial mechanics, and indirectly his proofs for universal gravitation. Consider the following aspects of the range of this proposition's applicability.

- **Any Force Law:** This proposition will apply with any force law. Beware of assuming now or later that this somehow depends on an inverse square force law. The forces could be completely independent of the distance of the body (as well as of any other property of the body). Or, if they depend on distance, they could be directly as the distance to some power. They could also vary with time since the creation of the universe or depend on the movement of sunspots or the population of tent caterpillars.

- **Changing Force:** The forces may be different from moment to moment not only in magnitude but also among positive, negative, and zero values. As long as the times are held equal, the triangles will be equal regardless of the values of the forces. Nothing is specified about the forces except that they are centripetal and directed to a center that is immobile. As shown in the proof, the forces can be positive, zero, or negative; they can be constant or varying.

- **Body at the Center:** Nothing is said about any body at the center of forces, and no such body is assumed. The center of forces is a mathematical entity, a geometrical point around which the equal areas may be found. It will be a key discovery in the development of Newton's theory of universal gravitation that in certain circumstances there *are* bodies at this geometrical center of forces.

We must be cautious here, as we always must as we work through this development, not to be assuming what we "know" as a consequence of this book's conclusions having become part of our current world view. Remember that as we go into *Principia* we understand gravity only as terrestrial heaviness, and what makes the heavenly bodies orbit is everybody's guess. (Section 3 of the Preliminaries gave what some of everybody's guesses were—it might be a good idea to review that to help you create or remain in the appropriate pre-Newtonian frame of mind.)

For more motivation on this, remember (or refer back to) the encouragements and warnings in Section 3 of the Preliminaries. Encourage yourself by thinking about the importance and excitement of discovering the true foundations for all our physical intuitions, and warn yourself about the frustrations to be encountered if we bring in our assumptions and then must deal with a Newton who seems to have been too stupid to see how obvious his proposition is, and so diabolically perverse as to insist on a long difficult proof for what we learned in third grade.

• **Center of Force, Force at the Center:** We might be inclined to think of the center as a center of *force* (singular) because we are imagining something there exerting that force. But Newton is presenting it as the point towards which all the forces around the path, the forces at each of the infinite numbers of points on the path, are directed, and is consistent in this enunciation, and in all the corollaries, in calling it "center of forces." This is not a consequence of his speaking of *bodies* (plural) in the enunciation, since in Corollaries 1–3 the body is singular.

We might even catch ourselves talking about "the force at the center." But Newton has not given us a force at the center, only a center towards which the forces are directed. On the other hand, during the course of the sketch, he does allow himself to speak about "the centripetal force" (singular) acting with an impulse, acting successively, and (at the limit) perpetually drawing back the body from the tangent to the curve.

The important thing is that none of Newton's proofs in any way assumes or depends upon the force being singular or being exerted from the center. This generality of the proofs has an important significance in the larger system of *Principia*. Newton says he does not know the cause of gravity, and that he contrives no hypotheses about it (see the General Scholium, page 488). Thus we don't know whether the force that turns the planets out of their tangential inertial path is a single force somehow pulling from the center, or many forces pushing from behind the planets, or something different yet. Because we don't know, we need our demonstrations to be general enough to allow for the different possibilities.

• **Orbit:** Although the word "orbit" may suggest a closed path to you, there is no assumption in the proof of this proposition that the path be closed.

• **No "ghost curve":** The bases of triangles in the proposition—the distances the body travels between impulses of force—do not circumscribe, and are not inscribed in, this ultimate curve, nor do they connect to it or follow it in any other way as a kind of "ghost curve." They are more like tangents to the curve-to-be than like chords; but they are not tangents, either, because the curve does not exist as long as the force is impulsive: it exists only at the limit. The points *A*, *B*, *C*,… of the finite case polygon do not necessarily fall anywhere on the ultimate curve, nor do any of the points on the sides such as *AB, BC, CD*.

• **Accelerative force.** Here and in general throughout Books I and II, when Newton says "force" he means accelerative quantity of force (Definition 7). It is a measure of force proportional to the velocity that is generated by that force in a given time. It is only in Book III that he will begin to work with motive quantity of force (Definition 8), which is measured by the motion generated in a given time, and thus involves mass.

Newton has a complex and difficult structure to build. He begins with the simpler case, one in which mass is ignored. Only when he has established what he can about accelerative force does he take the next step, which is bringing in the mass of the attracted body. This gives us another level of complication. Then, finally, he brings in the third level of complication, the mass of the attracting body. We may be grateful to him for building gradually in this way, since, had he not, what is already difficult would have been much more so. These second and third levels of complication are introduced in Book III.

Expansion of Newton's Sketch of I.1

"The areas which bodies driven in orbits [*gyros*] describe by radii drawn to an immobile center of forces, are contained in immobile planes and are proportional to the times."

Given:

Immobile center of forces.

To Prove:

1. The areas that bodies driven in orbits describe by radii drawn to that center of forces are proportional to the times;
2. the path of the body and the center of forces remain in the same plane.

Proof:

Part 1

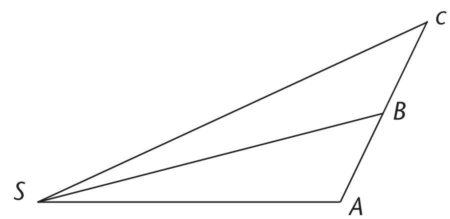
Step 1: Equal Areas in Equal Times

"Let the time be divided into equal parts, and in the first part of the time let the body, by its inherent force, describe the straight line AB. In the second part of the time, the same body, if nothing were to impede it, would pass on by means of a straight line to c (by Law 1), describing the line Bc equal to AB, with the result that, radii AS, BS, cS being drawn to the center, the areas ASB, BSc would come out equal."

The First Law of Motion stated:

> Every body continues in its state of resting or of moving uniformly in a straight line, except insofar as it is driven by impressed forces to alter its state.

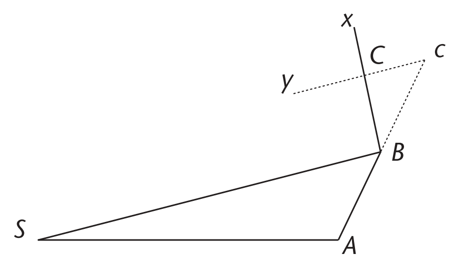
Suppose that in the first part of the time the body moves from *A* to *B* by its inherent force. That is, it has some velocity at *A* and continues with that same velocity in a straight line unless some added force changes it, according to the First Law of Motion. From A to B we are assuming no external force is operating on the body.



In the second part of the time it would, again by Law 1, if not hindered, move directly in a straight line to c, where $Bc = AB$ (since the times are equal and no force has been impressed to change the velocity).

By Euclid I.38 (same height, equal bases), $\triangle SAB = \triangle SBc$. Therefore in this case, the situation or instance of a zero force, equal areas will be described by the radii in equal times.

"But when the body comes to B, let the centripetal force act with an impulse that is single but great, and let it have the effect of making the body depart from the straight line Bc and continue in the straight line BC. Let cC be drawn parallel to BS, meeting BC at C; and, the second part of the time being completed, the body (by Corollary 1 of the Laws) will be located at C, in the same plane as the triangle ASB."



Now suppose a centripetal force acts on the body at B, turning it aside into line Bx. This force is to be understood as a single impulse of significant magnitude operating at that moment. That magnitude is measurable in the amount by which Bx is deflected from its default line Bc. To find that magnitude, draw cy from c parallel to SB, meeting Bx at C.

The centripetal force is impelling the body from B towards S. BS is its direction of force. Its magnitude is found by the actual deflection cC that brings the body, in a line parallel to BS, to the actual point C. C is reached in the same time that would have taken the body to c.

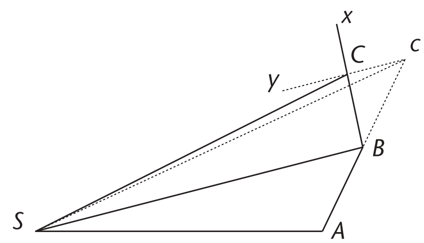
The times here for all these motions AB, Bc, BC, and so on, are equal; but the distances traveled AB, BC, and so on, are not necessarily equal.

We will call cC the effect of the force exerted as an impulse on the body at B for purposes of resolving the two forces.

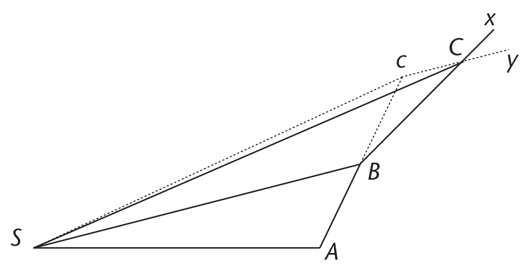
BC is the resolution of the first force Bc (where its innate force is impelling it) and the second force cC (what centripetal force has accomplished in moving it to C instead of c). BC is the body's actual motion in the given time.

Note that although Bc is necessarily equal to AB, BC is not necessarily the same length as Bc. One must find C using the parallel to line BS.

"Connect SC, and, because of the parallels SB, Cc, triangle SBC will be equal to triangle SBc, and therefore also to triangle SAB."



$\triangle SBc = \triangle SBC$ because they are on the same base SB and between the same parallels SB and Cc [Eu. I.37]. Therefore it will also be true that when we have a positive centripetal force, a nonzero force directed towards the center, equal areas will be described by the radii in equal times.



Suppose finally that we have a repulsive force directed away from the center of forces, a negative centripetal force. Bx will now lie on the other side of Bc, as will our new point C. However, triangles SBC and SBc will still lie on the same base between the same parallels and so the areas will be equal. Thus if we had a negative centripetal force, a nonzero force away from the center, we would also have equal areas described in equal times.

We will show in a moment that not only does C lie in the plane of $\triangle SAB$ but also the center of forces and all the triangles lie in one plane, as asserted in the enunciation.

"By a similar argument if the centripetal force should act successively at C, D, E, and so on, making the body describe the individual straight lines CD, DE, EF, and so on, in the individual particles of time, all these will lie in the same plane, and the triangle SCD will be equal to the triangle SBC, and SDE to SCD, and SEF to SDE. Therefore, in equal times equal areas are described in a motionless plane;..."

Now suppose this process to be repeated with always the same increments of time. In the equal-time increment (the "individual particle of time") the body will travel the resultant between the motion due to its inherent force and the motion it would have had under the influence of the centripetal force, in a way exactly analogous to our demonstration above. Thus the area of every triangle will remain equal, regardless of how the forces may vary.

Step 2: The Center of Forces and All the Triangles Lie in One Plane

Triangle *SAB* defines a plane, so those three points are in one plane [Eu. XI.2].

A body travelling in a straight line *ABc* will continue in the same plane, so line *Bc* is also in the "immobile plane" [Eu. XI.1].

Since *c* and *S* are both in the plane, line *cS* will be, and so will triangle *SBc*.

Since *B* is impelled toward *S* by the centripetal force, the motion from that force *cC* will be in the plane of *B* and *S*, namely the given plane. So *C* is in that plane, and triangle *SBC* also.

And so on for all successive changes of motion.

Step 3: Proportionality

"...and, *componendo*, any sums whatever of areas *SADS*, *SAFS* are to one another as are the times of description."

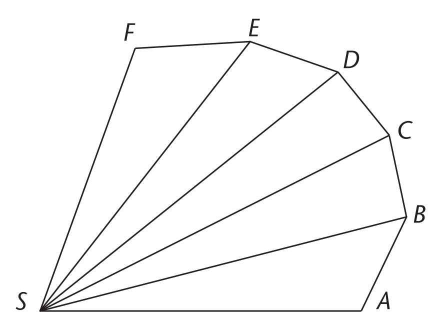
Having demonstrated the equal areas for equal times, we can add together any number of consecutive triangles (having proved in Step 2 that they all lie in one plane) to form polygonal areas; these areas will be proportional to the times of their description.

Part 2

"Now let the number of the triangles be increased and their breadth decreased *in infinitum*, and their ultimate perimeter *ADF* (by Corollary Four of Lemma Three) will be a curved line: and therefore the centripetal force, by which the body is perpetually drawn back from the tangent of this curve, will act without ceasing, while any described areas whatever *SADS*, *SAFS*, always proportional to the times of description, will be proportional to those same times in this case.

Q.E.D."

Since any described area (sum of triangles) is proportional to the time, this will also hold for evanescent triangles, the limiting case as the equal times approach zero. This limit of a sum of triangles, as in Lemma 3 Corollary 4, will be a curvilinear area. In this case as well, the areas will be proportional to the times. Q.E.D.



Note that as the time approaches zero, the forces, while in theory remaining impulsive, approach closer and closer to acting at every point. The limiting case, corresponding to an actual curvilinear path, may be thought of as a continuous force, although that might misleadingly suggest a single force. Perhaps we should say "forces acting continuously turning the body out of its tangential motion."

Pause After Proposition 1

• When we first encountered Definition 3, the definition of inherent force, we asked ourselves in what sense this was a force (or perhaps the question was what force meant to Newton if he included inertia as a force). One might be struck, therefore, to see him using both inherent force and impressed force to calculate the path of the body in this proposition; this seems to be an application in which their homologous effects would justify seeing them as the same sort of entity.

But do note that Newton is not actually using the two sorts of force to yield a resultant force; rather, he is letting each individual sort of force result in a motion and resolving the two motions into a resultant.

• Before going into the proposition we noted the many ways it was amazingly open. But there is one respect in which we might want to note that application of the proposition is limited: it does not claim that this method of reducing time intervals will lead to any curve you please. It is explicitly applied to a certain class of curves: those generated when a body is moving in response to forces applied along radii to a fixed center of forces.

• Note that Newton has not offered an argument that the limiting case in the proposition is a unique curve. This was not his goal. He has undertaken to prove that, *whatever* curve results, there will be equable description of areas and that the body and the center of forces remain coplanar. This is what he needs, and what he does in fact prove.

• In the *New Astronomy* (1609), Johannes Kepler invented the areas-proportional-to-times rule as a way of mathematizing the observed fact that planets go faster as they get closer to the sun. This later came to be known as Kepler's Second Law. Newton is here deriving the same conclusion starting from a hypothesis of central forces (forces directed radially towards or away from a fixed center). Its agreement with what had been recognized as the real-world applicability of Kepler's rule does not, however, consititute proof that Proposition 1 applies to our world. It is still only hypothetical, and will be applied to our world in Book III. At this point in the unfolding of *Principia,* we don't know whether there are such things as central forces in our world. What this proposition tells us is that if there are such forces operating, the result will be equable description of areas.

Notes on I.1 Corollaries

● The following are corollaries to the proposition; the arcs being invoked should be understood as arcs of the curve that is the limiting case of the proposition.

● Remember that in all these propositions and corollaries, it is important to note which assertions are of relationships true generally, and which are only true of the ultimate case as the arcs evanesce. Careless use of the latter for the former will lead to false conclusions. Every indication of relationship (such as $::$, $\sim$, $\propto$, $=$), when it applies only at the limit, needs to have expressed with it (that is, written before or over the relationship symbol) the condition "ult."

I.1 Corollary 1

The velocity of a body attracted to an immobile center in nonresisting spaces is inversely as the perpendicular dropped from that center to the rectilinear tangent of the orbit. For in those places A, B, C, D, E, the velocity is as the bases of the equal triangles AB, BC, CD, DE, EF, and these bases are inversely as the perpendiculars dropped to them.

[Note on translation: Though the wording of the last sentence may seem awkward, we have chosen to leave it as Newton wrote it, with velocity in the singular. To change this to a plural presumes to edit Newton and deprives us of what may be an insight into the way he was thinking about these matters.]

[Note on diagram: Newton gives no separate diagram for the corollaries. For Corollary 1, it is safe to use the diagram he gives with the proposition.]

Notes on I.1 Corollary 1

● This first corollary has not entirely left the impulse model. What the corollary proves is true for finite times as well as at the limit where the forces are exerted continuously, so we could use the diagram of the proposition.

● Kepler said the elapsed times over equal arcs vary directly as the distances *SA*, *SB*, etc. It would follow from this that the speeds were inversely as the distances *SA*, *SB*, etc. Newton is saying that Kepler's formulation isn't quite right, that the speeds are inversely as the perpendiculars to *AB*, *BC*, etc.

Does it strike you as odd that the speeds should depend, not on the distance to the body, but on the distance to some place where nothing is?

● In the proposition Newton spoke of bodies being driven in orbits and of centripetal forces acting with the effect that the body departs from the straight line of the tangent. These are neutral images. Here, however, he speaks of a body being "attracted" to an immobile center.

This language was the subject of a warning in Section 4.2.1 of the Preliminaries; review Newton's disclaimers quoted in that section if you feel you may be about to make some assumption about the cause or mechanism of these centripetal forces.

Expansion of Newton's Sketch of I.1 Corollary 1

Given:

1. Immobile center of forces;
2. curvilinear orbit formed by impulses as in proposition.

To Prove:

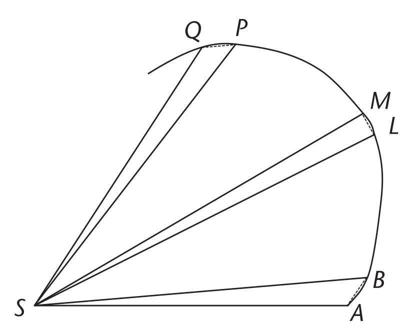
Velocities will be inversely as the perpendiculars to the tangents.

Proof:

We begin here with an orbit—a curvilinear path. We saw in the proposition that we can get to a curved path by letting the equal-time increment approach zero. This final figure of the proposition is the curved path made up of evanescent bases *AB*, *BC*, *CD*, etc.

Now let's approach the curved path another way. Suppose we are given a curved orbit, say the final limiting case curve of the proposition or maybe one given by observation of a body moving under forces directed to a center. Take equal-time arcs along this orbit, and construct the chords of those arcs.

Let's look at three of the equal-time arcs on the given curved path—arcs *AB*, *LM*, and *PQ*—and at the triangles on bases *AB*, *LM*, and *PQ*.



Since the arcs are described in equal times, they will be to one another in length as the average velocities along them. We can write this as:

arcs $\propto$ average velocities.

> **Note:** Remember that this "$\propto$" notation is shorthand for a proportion. Whenever we see it, we know that we are actually talking about at least four terms.
> So here, for example, we could expand this to:
>
> $$\frac{\overset{\frown}{AB}}{\overset{\frown}{LM}} = \frac{\text{av. vel. over arc } AB}{\text{av. vel. over arc } LM}, \qquad \frac{\overset{\frown}{LM}}{\overset{\frown}{PQ}} = \frac{\text{av. vel. over arc } LM}{\text{av. vel. over arc } PQ}, \text{ etc.}$$

But we're looking now at the limiting case as $\triangle t \to 0$. Here $B \to A$, $M \to L$, and $Q \to P$. Therefore the arcs AB, LM, etc. will be to one another as the instantaneous velocities at A, L, and P,

$$\frac{\widehat{AB}}{\widehat{LM}} \overset{ult}{=} \frac{\text{vel}_A}{\text{vel}_L}, \qquad \text{etc.}$$

But by Lemma 7, the chords are ultimately as the arcs.

$$\frac{\overline{AB}}{\overline{LM}} \overset{ult}{=} \frac{\text{vel}_A}{\text{vel}_L}, \quad \text{etc.}$$

The arcs, and therefore ultimately the chords, are the equal-time distances traveled.

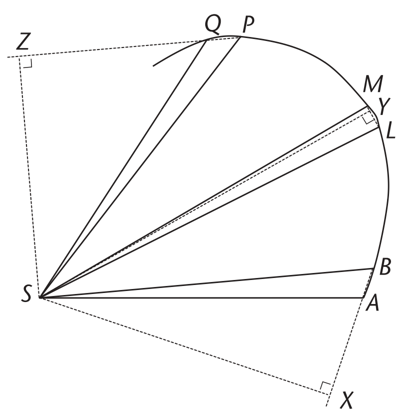
It can be proved starting from Euclid I.39 and Euclid VI.1 that bases of equal triangles are inversely as the heights of the triangles. (Find the heights of the triangles by dropping perpendiculars from S to the bases extended as necessary.)

So bases $\propto$ 1 / perpendiculars to chords.

Therefore velocities $\overset{ult}{\propto}$ 1 / perpendiculars to chords.

> **Note:** Remember that this "$\propto$" notation is shorthand for a proportion. Here, however, it is an inverse proportion—the second ratio is the inverse of the first. That is, for example:
>
> $$\frac{\text{vel}_A}{\text{vel}_L} \overset{ult}{=} \frac{SY}{SX} \qquad \text{or} \qquad \frac{\text{vel}_P}{\text{vel}_A} \overset{ult}{=} \frac{SX}{SZ}.$$

By Lemma 6, as the arcs evanesce so that $Q \to P$, $M \to L$, and $B \to A$, the limiting positions of the chords are the tangents.



$SZ \perp PQ$

$SY \perp LM$

$SX \perp BA$

Therefore the limiting positions of the perpendiculars to the chords are the perpendiculars to the tangents.

Thus the velocities at these points P, L, A will ultimately be inversely as the perpendiculars to the tangents to the evanescent arcs.

velocities $\overset{ult}{\propto}$ 1 / perpendiculars to tangents. Q.E.D.

I.1 Corollary 2

If the chords AB, BC of two arcs described successively in equal times by the same body in nonresisting spaces be completed into the parallelogram $ABCV$, and its diagonal, in that position which it ultimately has when those arcs are diminished in infinitum, *be produced in both directions, it will pass through the center of forces.*

Notes on I.1 Corollary 2

• The claims of this and the following corollaries can be recognized as true in the case of the polygonal areas set up in the main part of the proof of the proposition. But the enunciations will not talk about that case. They will assert certain things about geometric entities built in curved paths. We will look at perpendiculars to tangents to curved paths and parallelograms built on chords of arcs of such curves.

We will have to be very careful in these corollaries, and in Proposition 2, not to confuse the polygonal area enclosed by chords of the curved orbit with the polygonal area made up of the triangles of impulsive forces. They are not the same. Remember that the polygon of the impulsive model was not following a “ghost curve.” (See the last bullet in the notes to the proposition.)

Most of what is true of the finite case of the impulsive-model polygon is not true of the finite case of the polygon made of chords on arcs of a given curved path. So don’t let the fact that something has been proved (or is obvious) for the finite-time-increment impulse model mislead you into thinking it needn’t be proved for the orbit and its inscribed polygon.

• The figure made of these chords is a polygon inscribed in the curve. It is a different polygon from the one we constructed in the proposition, and the parallelograms such as $ABCV$ will be different.

If we reduce the equal-time increments for the arcs on the curved path, the perimeter of that polygon will approach the curve. Thus the same curve is the limit both of the polygon made of its chords and of the polygon of the proposition.

This means that as long as we deal with evanescent time increments, evanescent arcs and chords, and evanescent triangles, we may use either the figure of the proposition or a figure made of chords of equal-time arcs. Since very different things will be true of the two figures for finite time increments, we will have to proceed with care and make sure all our steps are justified. But this will be the key to our strategy in this and the following corollaries.

Expansion of Newton's Sketch of I.1 Corollary 2

Given:

1. Immobile center;
2. two arcs of a curved orbit described successively in equal times by a body moving under the influence of forces directed towards a center.

To Prove:

Diagonal of parallelogram completed out of chords of those two arcs will ultimately pass through the center of forces when the arcs evanesce.

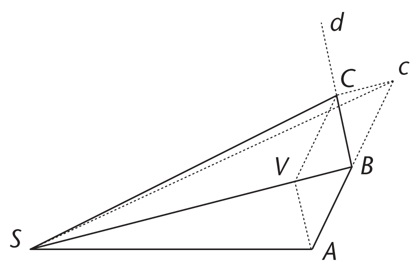


Figure 1

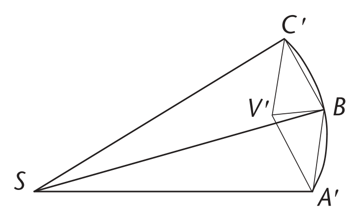


Figure 2

Proof:

Let's look at two situations now. The first, shown in Figure 1, is the polygon of the finite case in the proposition and shows two equal-time polygonal bases with a parallelogram *ABCV* constructed on them.

The second, shown in Figure 2, is the ultimate case of the proposition, where the triangles have evanesced and the path is a curve. This figure shows equal-time arcs with chords drawn across those arcs.

In Figure 1, $Bc = AB$ by the construction of the proposition; and therefore, by Euclid I.34, $Bc = CV$. Therefore diagonal *BV* of the parallelogram *ABCV* is the side of the parallelogram *BcCV* in Newton's diagram for the proposition, and $BV \parallel cC$.

By the proposition, a line from *B* parallel to *cC* will go through the center of forces. Therefore diagonal *BV* of the parallelogram *ABCV* will go through the center of forces. This is true by the construction of the proposition and applies always, not just in the limiting case.

Thus the diagonal of the parallelogram here will converge towards the center of forces, as asserted in the enunciation of this corollary. But this corollary is not talking about the finite case polygon of the proposition, but of the figure made from chords of equal-time arcs taken on the curved orbit of the ultimate case of the proposition.

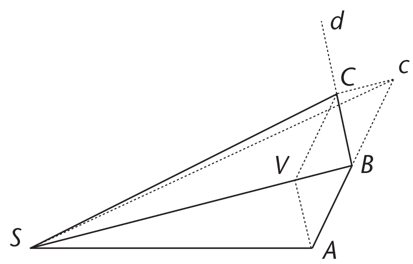


Figure 1

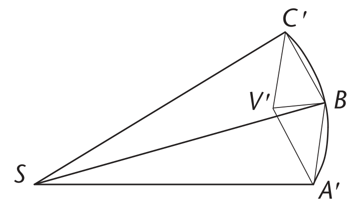


Figure 2

So let's look at the curved orbit in Figure 2. This is the ultimate case of the proposition, with equal-time arcs and chords of those arcs. Again we take two equal-time segments $A'B'$ and $B'C'$ and construct a parallelogram $A'B'C'V'$. In a polygon of chords there seems no reason for diagonal $B'V'$ to go through the center of forces S as the corollary asserts.

Here's how we prove that it does. Consider letting $\triangle t \to 0$ in both the polygonal path of the proposition's finite case (Figure 1) and the polygon of chords of arc (Figure 2). We saw in Part 2 of the proposition that the Figure 1 path approaches the curve of Figure 2. And by Lemma 7, the polygon of Figure 2 will also approach the curve.

As $\triangle t \to 0$, both polygons have the same limiting curve; therefore what is always true of the proposition's finite polygonal path will be *ultimately* true of the figure made by the chords of equal-time arcs on the orbit.

We showed that with the polygonal path of the proposition (Figure 1), the diagonal of the parallelogram BV, which is parallel and equal to cC, will always lie on line BS. Therefore it will do so in the limiting case of the curved orbit and its chords used in this corollary (Figure 2).

Q.E.D.

I.1 Corollary 3

If the chords AB, BC, and DE, EF, of arcs described in equal times in nonresisting spaces be completed into parallelograms ABCV, DEFZ, the forces at B and E are to each other in the ultimate ratio of the diagonals BV, EZ, where these arcs are diminished in infinitum. *For the motions BC and EF of the body are (by Corollary 1 of the Laws) composed of the motions Bc, BV, and Ef, EZ; and BV and EZ, equal to Cc and Ff, were in the demonstration of this proposition generated by the impulses of the centripetal force at B and E, and therefore are proportional to these impulses.*

Notes on I.1 Corollary 3

● This is not, properly speaking, a corollary of Proposition 1, and should, perhaps, have been presented as a separate proposition. It does not depend on Proposition 1. Rather, it is a simple application of Corollary 1 of the Laws of Motion to bodies and forces.

● Proposition 1 applies to a single orbit. Newton's use in Corollary 3 of sequential alphabetical letters for his orbital points suggests that he also has in mind a single orbit here (or maybe he just wanted to avoid having to commission another diagram).

However, nothing is assumed or asserted in this corollary about centers of forces or centripetal forces, and nothing in the logic assumes a single orbit. Furthermore, Newton uses this corollary in the proof of Corollary 4, which explicitly refers to different orbits.

We thus make a shift in Corollary 3 and now consider what may be different orbits and forces.

● This corollary, whose immediate applicability may seem obscure, lays a foundation that will be drawn upon first in Proposition 4, then further generalized in Proposition 6, and finally, in Proposition 6 Corollary 1, gathered into a powerful expression that Newton will use to determine the relationship of forces under many different force laws.

Expansion of Newton's Sketch of I.1 Corollary 3

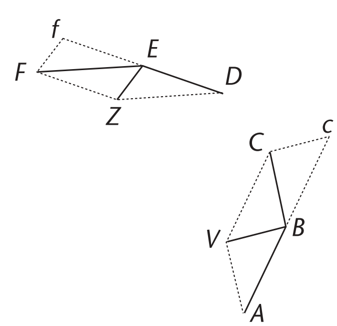
"If the chords AB, BC, and DE, EF, of arcs described in equal times in nonresisting spaces be completed into parallelograms $ABCV$, $DEFZ$, ..."

Given:

1. Immobile center of forces;

2. pairs of successive chords, all of which are of equal-time arcs;

3. nonresisting spaces.



To Prove:

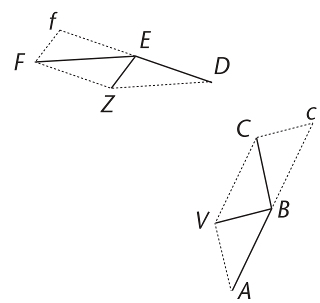
"... the forces at B and E are to each other in the ultimate ratio of the diagonals BV, EZ, where these arcs are diminished *in infinitum*."

The diagonals of parallelograms made from each pair of successive chords will be to one another ultimately as the forces at the intersection of the two chords.

Proof:

We are supposing evanescent parallelograms with impulsive forces being added at B and E. We suppose further that in the same evanescent time that the body actually goes to C and F, it would have gone to c and f.

"For the motions BC and EF of the body are (by Corollary 1 of the Laws) composed of the motions Bc, BV, and Ef, EZ; ..."



By Corollary 1 of the Laws of Motion, the actual paths BC and EF are composed of the sides of the parallelogram Bc, BV, and Ef, EZ.

"... and BV and EZ, equal to Cc and Ff, were in the demonstration of this proposition generated by the impulses of the centripetal force at B and E, and therefore are proportional to these impulses."

As in the construction for Proposition 1, the magnitudes of force generated by impulses at B and E are represented by BV and EZ. The length of the lines representing these forces can be found by the lengths of cC and fF, which measure the actual deflection. By Euclid I.34, in parallelogram $FfEZ$, $fF = EZ$ and in parallelogram $CcBV$, $cC = BV$.

Now we consider the path AB taken by the body in the same evanescent time period before the impulse was added at B. By Law 1, $Bc = AB$. But by Euclid I.34, $Bc = VC$. Therefore $AB = VC$. Because BC continues in straight line AB, $AB \parallel VC$. Thus by Euclid I.33, area $ABCV$ is a parallelogram, with BV its diagonal.

Similarly, EZ is the diagonal of parallelogram $DEFZ$.

Thus the forces at B and E are to each other in the ultimate ratio of the diagonals BV, EZ, where arcs $\widehat{ABC}$ and $\widehat{DEF}$ are diminished *in infinitum*.

Q.E.D.

I.1 Corollary 4

The forces by which any bodies in nonresisting spaces are drawn back from rectilinear motions and are deflected into curved orbits are to one another as those sagittae of arcs described in equal times which converge to the center of forces and bisect the chords, when those arcs are *diminished* in infinitum. *For these sagittae are the halves of the diagonals with which we have been concerned in Corollary 3.*

Notes on I.1 Corollary 4

• Corollary 4, which extends Corollary 3, is also applicable to different orbits, as Newton makes explicit by referring to "any bodies" (plural). It requires only that each orbit has a center towards which its own forces are directed. In each individual case, Proposition 1 Corollary 2 is called upon to note that the crucial diagonals go through the respective centers of forces.

For future applications, we should note that the logic of the argument in no way assumes or depends upon there being a jointly shared center. In fact, Newton himself invokes this corollary in Proposition 4 to apply to a case where there are different orbits around different centers of forces.

However, because Newton's wording makes "center" singular, the diagram provided here shows a shared center for both orbits. This case, which is also a possible application of the corollary, is the one found when several planets orbit the sun as a shared center of forces.

• As in Lemma 11, the sagittae are lines that converge to a point, here the center of forces, and bisect the chords. The segment that is the actual sagitta is the part between the chord and the curve.

Expansion of Newton's Sketch of I.1 Corollary 4

Given:

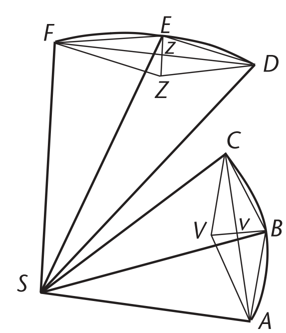
1. Immobile center of forces;
2. nonresisting spaces;
3. equal-time arcs of curved orbits.

To Prove:

Centripetal forces ultimately vary as the sagittae of the equal-time arcs.

That is, $f_1 : f_2 \overset{ult}{::}$ sagitta$_1$: sagitta$_2$

or $f \overset{ult}{\propto}$ sagittae.

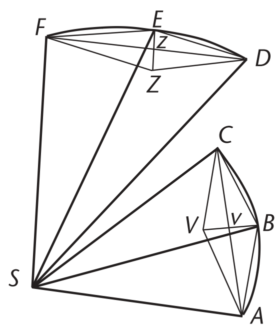


Proof:

To keep the same conditions as Corollary 3, let arcs *AB*, *BC*, *DE*, and *EF* be arcs described in equal times. Arcs *AC* and *DF* are also equal-time arcs, with times doubled. Draw chords *AC*, *DF*.

Note: We can draw lines from point S, bisecting the chords AC and DF, but they might not cut the curve at B and E. Alternatively we can connect SB and SE, but those lines may not bisect the chords AC, DF. In fact, for curves other than circles with the center of forces at the center, lines SB and SE would generally not bisect the chords in the finite case.

Complete the parallelograms $FEDZ$ and $CBAV$, in each of which two of the original equal-time chords form two sides. Diagonals AC, BV and DF, EZ bisect each other (prove using Euclid I.34, I.29, I.15, I.26). Therefore Bv, Ez are the lines that actually bisect the chords AC and DF.



We set up our equal-time arcs of the curved path, and, as in the previous two corollaries, we made parallelograms on chords of two pairs of successive equal-time arcs such as $\widehat{AB}$, $\widehat{BC}$ and $\widehat{DE}$, $\widehat{EF}$. One diagonal of each of those parallelograms constitutes the chord of the two successive equal-time arcs; they will be chords of new double-length equal-time arcs such as $\widehat{AC}$, $\widehat{DF}$. These are the equal-time arcs to which the sagittae mentioned in the corollary apply.

Since by Corollary 2 the diagonals BV, EZ converge towards the center of forces only at the limit, consider now the ultimate case of the orbit and chords as $\triangle t \to 0$.

Bv ultimately lies on SB, and Ez ultimately lies on ES, by Corollary 2.

And $BV : EZ \overset{ult}{::} f_B : f_E$ by Corollary 3.

Since Bv, Ez (the ultimate sagittae) are half BV, EZ,

$Bv : Ez \overset{ult}{::} f_B : f_E$.

That is, forces that draw bodies out of rectilinear motion into a curved orbit are ultimately as the sagittae of arcs traversed in equal times.

Q.E.D.

I.1 Corollary 5

Therefore, the same forces are to the force of gravity [gravitas] *as these sagittae are to the sagittae, perpendicular to the horizon, of parabolic arcs which projectiles describe in the same time.*

Notes on I.1 Corollary 5

This corollary is not used in this basic sequence of propositions; nevertheless some comments will be helpful in placing it in its context.

• Corollary 5 (not present in the first edition) was evidently included because it struck Newton that the groundwork for some of what he would develop later had already been laid here. It is a look ahead at how this proposition (I.1) is going to be applied in I.4 Corollary 9 and in III.4.

• The corollary is saying that the deflections under the respective forces, geometrically the sagittae, are a measure of the force, both in the case of a body in orbit and a cannonball. The term *gravitas*, it must be remembered, means only terrestrial heaviness, and so far may only be applied to the cannonball.

But in the case of both curves, the curve of the body in orbit and curve of the cannonball in its parabola, there is something geometrically common. The deflection that the centripetal force produces and that which the force of gravity produces are both deflections from the tangential inertial path; and this deflection can be measured in both cases by the sagittae.

• However, nothing is being said here about the nature of the forces; it is only a geometrical observation. Thus it would be premature to conclude the cannonball and the heavenly body are being deflected by *the same* force; it would certainly be false to think Newton believed he had proved any such thing by this point.

I.1 Corollary 6

All the same things pertain, by Corollary 5 of the Laws, where the planes in which the bodies move, along with the centers of forces located in them, are not at rest, but move uniformly in a straight line.

Note on I.1 Corollary 6

Proposition 1 specifies that the center of forces is immobile; this corollary extends the proposition to apply to the situation when the whole plane is moving with uniform rectilinear motion. By Corollary 5 of the Laws,

> The motions of bodies contained in a given space are the same among themselves whether that space be at rest or move uniformly in a straight line without circular motion.