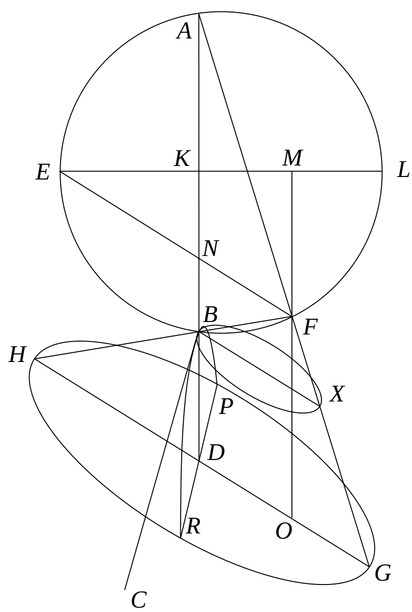
PROPOSITION 54 (PROBLEM)

Given two bounded straight lines perpendicular to each other, one of them being produced on the side of the right angle, to find on the straight line produced the section of a cone called hyperbola in the same plane with the straight lines, so that the straight line produced is a diameter of the section and the point at the angle is the vertex, and where whatever straight line is dropped from the section to the diameter, making an angle equal to a given angle, will equal in square the rectangle applied to the other straight line having as breadth the straight line cut off by the dropped straight line beginning with the vertex and projecting beyond by a figure similar and similarly situated to that contained by the original straight lines.

Let there be the two bounded straight lines *AB* and *BC* perpendicular to each other, and let *AB* be produced to *D*; it is required then to find in the plane through the lines *AB*, *BC* an hyperbola whose diameter will be the straight line *ABD* and vertex *B*, and upright side the straight line *BC*, and where the straight lines dropped from the section to *BD* at the given angle will equal in square the rectangles applied to *BC* having as breadths the straight lines cut off by them from *B* and projecting beyond by a figure similar and similarly situated to the rectangle *AB*, *BC*.



First let the given angle be a right angle, and on *AB* let a plane be erected at right angles to the plane of reference, and let the circle *AEBF* be described in it about *AB*, so that the segment of the circle's diameter within the sector *AEB* has to the segment of the diameter within the sector *AFB* a ratio not greater than that of *AB* to *BC,** and let *AEB* be bisected at *E,* and let the straight line *EK* be drawn perpendicular from *E* to the straight line *AB* and

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* Eutocius, commenting, adds: "Let there be two straight lines *AB* and *BC,* and let it be required to describe a circle on *AB* so that its diameter is cut by *AB* in such a way that the part of it on the side of *C* has to the remainder a ratio not greater than that of *AB* to *BC.*

"Now let it be supposed that they have the same ratio, and let *AB* be bisected at *D,* and through it let the straight line *EDF* be drawn perpendicular to *AB,* and let it be contrived that *AB* : *BC* :: *ED* : *DF,*
and let *EF* be bisected; then it is clear that if

$$AB = BC$$

and

$$ED = DF,$$

the point *D* will be the midpoint of *EF,* and if

$$AB > BC$$

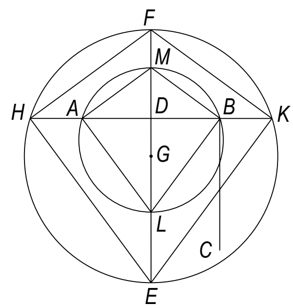
and

$$ED > DF,$$

the midpoint will be below *D,* and if

$$AB < BC,$$

it will be above *D.*



"And now [assuming *AB* > *BC*] let it be below as *G,* and with center *G* and radius *GF* let a circle be described; then it will have to pass either within or without the points *A* and *B.* And if it should pass through the points *A* and *B,* what was enjoined would be done; but let it fall beyond the points *A* and *B,* and let the straight line *AB,* produced both ways, meet the circumference at *H* and *K,* and let *FH, HE, EK* and *KF* be joined, and let *MB* be drawn through *B* parallel to *FK,* and *BL* parallel to *KE,* and let *MA* and *AL* be joined; then these will also be parallel to *FH* and *HE* because

$$AD = DB$$

and

$$DH = DK$$

and *FDE* is perpendicular to *HK.* And since the angle at *K* is a right angle, and *MB* and *BL* are parallel to *FK* and *KE,* therefore the angle at *B* is a right angle; then for the same reasons also the angle at *A.* And so the circle described on *ML* will pass through the points *A* and *B* (Eucl. III. 31). Let the circle *MALB* be described. And since *MB* is parallel to *FK,*

$$FD : DM :: KD : DB.$$

Then likewise also

$$KD : DB :: ED : DL.$$

And therefore

$$FD : DM :: ED : DL.$$

And alternately,

$$ED : DF :: AB : BC :: LD : DM.$$

"And likewise if the circle described on *FE* cuts *AB,* the same thing could be shown." (Tr.)

let it be produced to *L*; therefore the straight line *EL* is a diameter (Eucl. III. 1). If then

$$AB : BC :: EK : KL$$

we use point *L*, but if not, let it be contrived (Eucl. VI. 12) that

$$AB : BC :: EK : KM$$

with

$$KM < KL,$$

and through *M* let *MF* be drawn parallel to *AB*, and let *AF*, *EF*, and *FB* be joined, and through *B* let *BX* be drawn parallel to *FE*. Since then

angle AFE = angle *EFB*,

but

angle AFE = angle *AXB*,

and

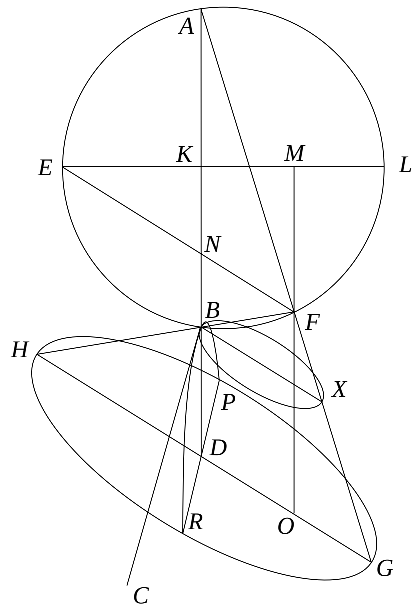
angle *EFB* = angle *XBF*,

therefore also

angle XBF = angle *FXB*;

therefore also

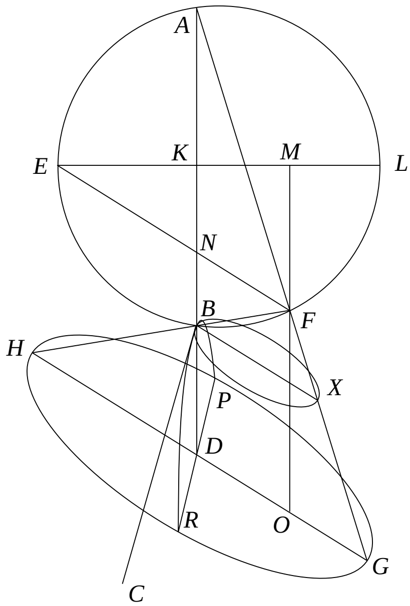
$$FB = FX.$$



Let a cone be conceived whose vertex is the point *F* and whose base is the circle about diameter *BX* at right angles to triangle *BFX*. Then the cone will be a right cone; for

$$FB = FX.$$

Then let the straight lines *BF*, *FX*, and *MF* be produced, and let the cone be cut by a plane parallel to the circle *BX*; then the section will be a circle (I. 4). Let it be the circle *GPR*; and so *GH* will be the diameter of the circle (I. 4, end). And let the straight line *PDR* be the common section of circle *GH* and of the plane of reference; then *PDR* will be perpendicular to both of the straight lines *GH* and *DB*; for both of the circles *XB* and *HG* are perpendicular to triangle *FGH*, and the plane of reference is perpendicular to triangle *FGH*; and therefore their common section, the straight line *PDR*, is perpendicular to triangle *FGH*; therefore it makes right angles also with all



the straight lines touching it and in the same plane.

And since a cone whose base is circle *GH* and vertex *F*, has been cut by a plane perpendicular to triangle *FGH*, and has also been cut by another plane, the plane of reference, in the straight line *PDR* perpendicular to the straight line *GDH*, and the common section of the plane of reference and of triangle *GFH*, that is the straight line *DB*, produced in the direction of *B*, meets the straight line *GF* at *A*, therefore by things already shown before (I. 12) the section *PBR* will be an hyperbola whose vertex is the point *B*, and where the straight lines dropped ordinatewise to *BD* will be dropped at a right angle; for they are parallel to straight line *PDR*. And since

$$AB : BC :: EK : KM,$$

and

$$EK : KM :: EN : NF :: \text{rect. } EN, NF : \text{sq. } NF,$$

therefore

$$AB : BC :: \text{rect. } EN, NF : \text{sq. } NF.$$

And

$$\text{rect. } EN, NF = \text{rect. } AN, NB \text{ (Eucl. III. 35);}$$

therefore

$$AB : CB :: \text{rect. } AN, NB : \text{sq. } NF.$$

But

$$\text{rect. } AN, NB : \text{sq. } NF :: AN : NF \text{ comp. } BN : NF;$$

but

$$AN : NF :: AD : DG :: FO : OG,$$

and

$$BN : NF :: FO : OH;$$

therefore

$$AB : BC :: FO : OG \text{ comp. } FO : OH,$$

that is

$$\text{sq. } FO : \text{rect. } OG, OH.$$

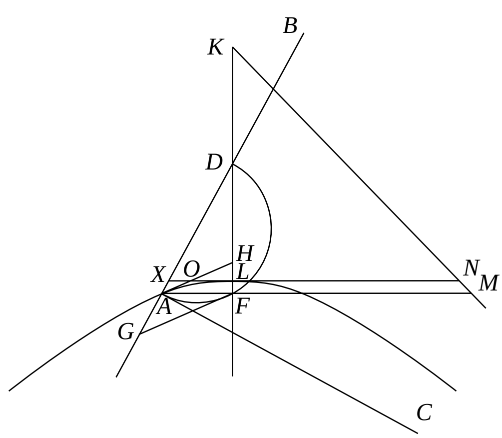
Therefore

$$AB : BC :: \text{sq. } FO : \text{rect. } OG, OH.$$

And the straight line *FO* is parallel to the straight line *AD*; therefore the straight line *AB* is the transverse side, and *BC* the upright side; for these things have been shown in the twelfth theorem (I. 12).

PROPOSITION 55 (PROBLEM)

Then let the given angle not be a right angle, and let there be the two given straight lines *AB* and *AC*, and let the given angle be equal to angle *BAH*; then it is required to describe an hyperbola whose diameter will be the straight line *AB*, and upright side *AC*, and where the ordinates will be dropped at angle *HAB*.



Let the straight line *AB* be bisected at *D*, and let the semicircle *AFD* be described on *AD*, and let some straight line *FG*, parallel to *AH*, be drawn to the